Math 246C Lecture 19 Notes

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1 Hartogs' Theorem

1.1 Lemmas containing the argument

The goal is to prove the following theorem.

Theorem 1.1 (Hartogs)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u : \Omega \to \mathbb{C}$ be separately holomorphic. Then $u \in \mathrm{Hol}(\Omega)$.*

We will break up the proof into a few lemmas.

Lemma 1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let u be separately holomorphic in Ω. If u is locally bounded in Ω, then $u \in C(\Omega)$ (so $u \in \mathrm{Hol}(\Omega)$).*

Proof. Let D be a polydisc with $\overline{D} \subseteq \Omega$. Write $D = D_1 \times D'$, where D_1 is a disc in $\mathbb{C}$ and D' is a polydisc in $\mathbb{C}^{n-1}$. The function $z_1 \mapsto u(z_1, z') \in \mathrm{Hol}(D_1)$. By Cauchy's integral formula, $\partial_{z_1} u(z_1, z')$ is bounded when $z_1 \in D_1' \subseteq D_1$ (compactly contained) and $z' \in D'$. It follows that $\partial_{z_j} u$ is bounded on a relatively compact polydisc $\subseteq D$; in other words, $\partial_{z_j} u$ are locally bounded in Ω. Also, $\partial_{\overline{z}_j} = 0$ for all j.

It follows that u is continuous. If $a \in \Omega$ and $h \in \mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$u(a+h) - u(a) = \sum_{j=1}^{2n} u(a + v_j) - u(a + v_{j-1}), \qquad v_j = (h_j, \dots, h_j, 0, \dots, 0).$$

Now use the mean value theorem. □

Induction on n: Now assume that Hartogs' theorem is already known for functions of $< n$ complex variables.

Lemma 1.2. *Let $u : \Omega \to \mathbb{C}$ be separately holomorphic, and let $D = \prod_{j=1}^n D_j$ be a closed polydisc $\subseteq \Omega$ with $D^o \neq \varnothing$. Then there exist discs $D_j' \subseteq D_j$ for $1 \leq j \leq n-1$ with nonempty interior such that if $D_n' = D_n$, then u is bounded on $D' = \prod_{j=1}^n D_j'$.*

Proof. Let $E_M = \{z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \;\forall z_n \in D_n\}$. E_M is closed: by the inductive hypothesis, $z' \mapsto u(z', z_n)$ is holomorphic in a neighborhood of $\prod_{j=1}^{n-1} D_j$ for each z_n and thus continuous; so

$$E_M = \bigcap_{z_n \in D_n} \left\{ z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \right\}$$

is an intersection of closed sets. Also, $\bigcup_{M=1}^{\infty} E_M = \prod_{j=1}^{n-1} D_j$: $z_n \mapsto u(z', z_n)$ is holomorphic near D_n for all $z' \in \prod_{j=1}^{n-1}$ and is thus bounded on D_n: $|u(z', z_n)| \leq M$ for $z_n \in D_n$.

$\prod_{j=1}^{n-1} D_j$ is a complete metric space, so by Baire's theorem, so E_M has nonempty interior for some M. So E_M contains a polydisc $D' = \prod_{j=1}^{n-1} D_j'$ with nonempty interior such that if $D_n' = D_n$, u is bounded in $D' = \prod_{j=1}^{n} D_j' \subseteq D'$. □

Lemma 1.3. *Let D be a polydisc $\{|z_j - z_j| < R : j = 1, \ldots, n\}$. Let $u : D \to \mathbb{C}$ be holomorphic in $z' = (z_1, \ldots, z_{n-1})$ for every fixed z_n, and assume that u is holomorphic and bounded in D' given by $|z_j - z_j^o| < r$ for all $1 \leq j \leq n-1$ for some $r > 0$ and $|z_n - z_n^o| < R$. Then $u \in \mathrm{Hol}(D)$.*

Proof. We may assume that $z^o = 0$. Take $0 < R_1 < R_2 < R$. Taylor expand $z' \mapsto u(z', z_n)$:

$$u(z', z_n) = \sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^{\alpha'}, \qquad |z_j| < R, 1 \leq j \leq n-1, |z_n| < R.$$

We have that

$$a_{\alpha'}(z_n) = \frac{\partial^{\alpha'}(0, z_n)}{(\alpha')!}$$

is holomorphic in $|z_n| < R$. This series converges normally in $|z_j| < R$ for $1 \leq j \leq n-1$. So $a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0$ as $|\alpha'| \to \infty$ for each z_n. Now we have that $|u| \leq M$ in D'. By Cauchy's estimates in z', we know that

$$|a_{\alpha'}(z_n)| \leq \frac{M}{r^{|\alpha'|}} \qquad \forall \alpha'.$$

Consider the sequence of subharmonic (in $|z_n| < R$) functions

$$\varphi_{\alpha'}(z_n) = \frac{1}{|\alpha'|} \log |a_{\alpha'}(z_n)|, \qquad |\alpha'| = \alpha_1 + \cdots \alpha_{n-1}.$$

Our bound gives us that $\varphi_{\alpha'}$ is uniformly bounded above in $|z_n| < R$. Since $a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0$ as $|\alpha'| \to \infty$,

$$\limsup_{|\alpha'| \to \infty} \varphi_{\alpha'}(z_n) \leq \log(1/R_2)$$

for all z_n. By Hartogs' lemma on subharmonic functions, if $|z_n| \leq R_n$, then for any $\varepsilon > 0$,

$$\varphi_{\alpha'}(z_n) \leq \log(1/R_2) + \varepsilon \leq \log(1/R_1)$$

for large $|\alpha'|$. In other words, for large $|\alpha'|$ and $|z_n| \leq R_2$,

$$|a_{\alpha'}(z_n)| R_1^{|\alpha_1|} \leq 1.$$

The series $\sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^{\alpha}$ converges absolutely for $|z_n| < R_2$ and $|z_j| < R_1$ (for all $1 \leq j \leq n-1$) and hence normally in D. So $u \in \mathrm{Hol}(D)$ as a limit of holomorphic functions (the partial sums). $\square$

1.2 Proof of the theorem from the lemmas

We can now prove Hartogs' theorem.

Proof. Let $z^0 \in \Omega$, and take a closed polydisc $\{|z_j - z_j^0| < 2R, 1 \leq j \leq n\}$. Apply the second lemma to the closed polydisc with $|z_j - z_j^0| \leq R$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| \leq 2R$. Then we get a polydisc of the form $|z_j - \zeta_j^0| < r$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| < R$ with $\{|z_j - \zeta_j^0| < r\} \subseteq \{|z_j - z_j^0| < R, 1 \leq j \leq n-1\}$ such that u is holomorphic and bounded there. In particular, $|z_j - z_j^0|$. In particular, $|\zeta_j^0 - z_j^0| < R$.

Consider the polydisc D given by $|z_j - \zeta_j^0| < R$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| < R$ (closure in Ω): in the polydisc, u is holomorphic in z' if z_n is fixed, and u is holomorphic and bounded in the polydisc $|z_j - \zeta_j^0| < r$ for $j = 1, \ldots, n$ and $|z_n - z_n^0| < R$. By the third lemma, u is holomorphic in D, which is a neighborhood of z_0. $\square$